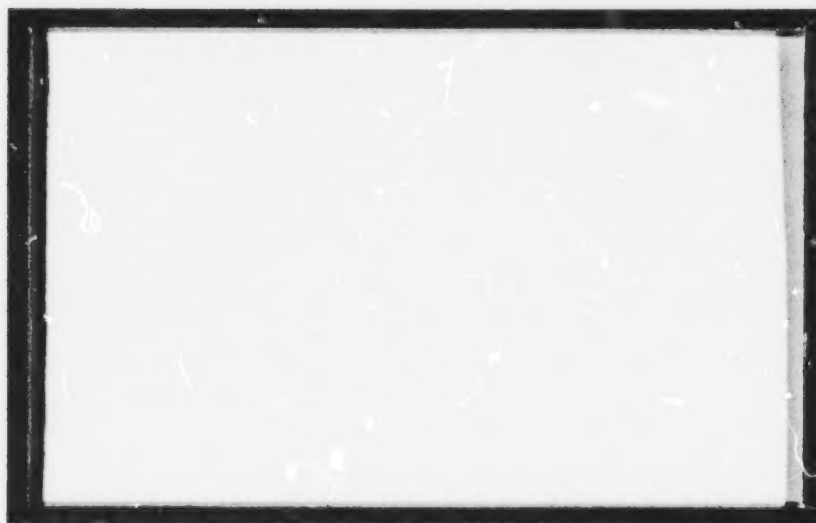


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MAPUM
A MODEL FOR ASSIGNING PASSENGERS
TO A FLIGHT SCHEDULE

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ABSTRACT

The MAPUM model determines passenger assignments given a daily flight schedule and the corresponding demand curve by time of day for each origin-destination pair. The objective function, to be minimized, incorporates an "attractiveness factor" and an "availability factor" to account for the aircraft capacity limitations. The constraints require each passenger to be assigned to a flight. The model presents a new approach to passenger assignment based on the concept of "passenger overload" and mathematical programming techniques are used in its solution.

1. The Problem

An airline has to forecast its schedule and its fleet to satisfy the growing demand and to be competitive with other airlines and other transportation modes. The model described in this paper is to be used in such a planning mode. The problem can be specified as follows: given a daily flight schedule and the corresponding demand curve by time of day for each origin-destination pair, assign passengers to flights. This passenger assignment model MAPUM (Model for the Assignment of Passengers developed at the Universit  de Montr al) can be used independently to evaluate a proposed flight schedule or could be integrated into an automatic schedule generator as presented in Soumis (1980).

Before defining the problem more explicitly, some basic terms are introduced. A *flight* is specified by its itinerary (Montr al - Chicago - Los Angeles), its departure and arrival times, and the aircraft type used. A *flight-leg* is a non-stop component of a flight (Montreal - Chicago). A *route* is composed of flight-legs not necessarily belonging to only one flight. For instance, a route from Montreal to San Francisco may include the flight-leg Montreal - Chicago (from the flight Montreal - Chicago - Los Angeles) and the flight-leg Chicago - San Francisco (from the flight Toronto - Chicago - San Francisco).

Two main factors determine the passenger's choice of a route: its attractiveness and its availability. The attractiveness of a route is a function of the time difference between the passenger's desired departure time and the scheduled departure time of the route, the time duration of the route (number of stops and/or connections), and the convenience

offered by the route (meals, wide body aircraft, etc.). The availability of a route is dependent on the aircraft capacity and the demand.

This is not an easy problem to solve mainly because the daily demand of each market is highly variable and several origin-destination pairs using the same flight-leg have to share the seats available. If the aircraft capacities are introduced explicitly as constraints in the model, then the problem becomes a multicommodity flow problem with bounds on the total flow in the arcs. This problem is very difficult to solve (See Kennington, 1978). This is the reason why most models incorporate the availability factor into the attractiveness factor. The proposed model is a compromise between these two extremes, and it reproduces reality more adequately since the variance in the daily demand is incorporated. The concept of "passengers overload" (PO) is used to integrate the capacity constraints into the objective function. The mathematical programming problem generated is easier to solve than a multicommodity network flow problem.

In Section 2, some approaches proposed to handle this problem are contrasted with the PO approach. Section 3 is devoted to an analysis of the attractiveness factor and the availability factor. An appropriate objective function is defined. In Section 4, the solution technique is summarized. A decomposition of the problem generates a sequence of subproblems that are very easy to solve. Some numerical results obtained with data from Air Canada are reported in Section 5.

2. Approaches to the problem

Several passenger assignment models have been proposed. They can be classified into three types: heuristic methods, simulation and mathematical programming models. This section does not give an exhaustive survey of all the models found in the literature. The aim is to briefly illustrate the different types. The emphasis is placed on mathematical programming methods since our model falls into this category and the literature on these models is more widely available. Most airlines have their own in-house models, but corporate restrictions limit publications of their research.

The most widely used models are heuristic. Sometimes, the approach is sequential (see Koyama, 1974). Each flight is filled sequentially, and the overflow of passengers is assigned to other flights. Other models (see Malik, 1976) assign passengers proportionally to flights. For some models (see Kingsley, 1968) the availability factor is incorporated through a measure of the attractiveness of different aircraft sizes. Air Canada developed such a model (see Gagnon, 1967).

Simulation can also be used to evaluate a flight schedule. Using historical data, demand for each flight can be generated randomly. An example of such a model is given by Lauchli (1976).

In some mathematical programming models, the availability factor appears as a group of explicit constraints. Several of these models have been developed in research centers, and many of these are integrated into larger systems to determine optimal frequency plan (see Agin, 1972, 1974;

Daigle, 1976; Richardson, 1976; Scarbrough, 1977; Swan, 1972). The passenger assignment problem can be defined as a linear programming problem where the objective is to maximize the total satisfaction of the passengers:

$$\text{Max} \sum_{i,j,p,q} w_{ij}^{pq} x_{ij}^{pq}$$

subject to

(2.1)

Flow conservation constraints for origin-destination pair p,q having a demand equal to d^{pq} .

for all (p,q)

(2.2)

Total flow on each arc (i,j) less than or equal to the capacity of the aircraft multiplied by the load factor

for all (i,j)

(2.3)

$$x_{ij}^{pq} \geq 0$$

for all i,j and for all p,q

where x_{ij}^{pq} is the number of passengers of origin-destination pair p,q on arc (i,j) ;

w_{ij}^{pq} is the satisfaction coefficient for passengers of origin-destination pair p,q using arc (i,j) .

In a variant of this model (see Currie, Dickey and Price, 1975), the number of passengers is defined in terms of routes instead of arcs because several origin-destination pairs are serviced by only a few routes. This modification generates a model with fewer variables and with the

following objective function:

$$\text{Max} \sum_{r,p,q} w_r^{pq} x_r^{pq} .$$

Constraints (2.1) are replaced by:

$$\sum_r x_r^{pq} = d^{pq} \quad \text{for all } p,q.$$

Accordingly, w_r^{pq} is the satisfaction coefficient for passengers of origin-destination pair p,q , using route r , and the satisfaction coefficient is taken as the weighted sum of the number of stops and connections. Even with this modification, a passenger assignment problem for 30 cities has approximately 750 constraints and 4 000 variables.

This model is a large scale linear programming problem. The multicommodity constraints (2.2) makes the problem difficult to solve in spite of its simple network structure. Agin (1972) and Scarbrough (1977) use the simplex method to solve it, but Agin's method has been tested only in a very special situation (see Agin, 1974) and Scarbrough has solved only small problems. Richardson (1976) uses the G.U.B. (Generalized Upper Bounded technique) approach to handle the multicommodity constraints. Unfortunately, computation time increases rapidly with the size of problem. This is consistent with results in Kennington (1978) comparing the G.U.B. approach, the Gradient approach and the Dantzig-Wolfe approach to solve these types of problems.

The linear models also have a tendency to generate extreme solutions where some aircraft are used almost to their limit while others are empty. To obviate this drawback and obtain more balanced solution, our model is non-linear. The approach is similar to that used for the traffic

assignment problem (see Florian, 1974; Nguyen, 1974).

Finally, we should note that other authors have introduced concepts similar to the ones used in our model. Boile (1974), Eriksen (1975), Eriksen (1977) define an attractiveness factor based on the principle that "time is money". Hence to each route they assign a cost which is a function of the time difference between the desired departure time and the route departure time, and the time difference between the travel time using this route and the travel time for a direct flight. Also the concept of *passenger overload* is similar to the concept of *spill* recently introduced in Lankford (1977) to analyse and forecast the surplus of seats to be offered at discount rates.

3. The model

A non-linear programming model is used to solve the passenger assignment problem. The objective function, to be minimized, includes two factors. The first factor is associated with the relative attractiveness of the routes. The second factor estimates the loss of revenue due to the expected number of passengers in excess of the aircraft capacity. The constraints require each passenger to be assigned to a route.

3.1. Attractiveness factor

The attractiveness of a route is a function of the departure time, the number of stops and/or connections, the aircraft type and the service quality. The interaction of these factors should be analysed by experienced airline personnel, and this is not within the scope of this paper.

Hence, as a first approximation, a simple linear term is used to incorporate this factor into the objective function. This term could be improved, and its expression could be as complicated as desired without altering the effectiveness of the model.

Assume that u is the scheduled departure time for route r and that t is the passenger's desired departure time. The passenger's dissatisfaction is evaluated as follows:

$$(3.1) \quad p(t,r) = a_1(t) + a_2(v)$$

where $a_1(t)$ is a penalty for the time difference between the flight time and the passengers' desired departure time and $a_2(v)$ evaluates the increase in flying time due to stops and/or connections.

$$a_1(t) = \begin{cases} k_1(u - t) & \text{if } u > t \\ k_2(t - u) & \text{if } t > u \\ 0 & \text{if } t = u \end{cases}$$

where k_1 is the penalty factor for the passengers' waiting time, and k_2 is the penalty factor for a passenger departing earlier than desired.

$$a_2(v) = k_3(v - \bar{t})$$

where v is the actual route time for the suggested route and $\bar{t}$ is the time for a direct flight. This component is a measure of the inconvenience due to stops and/or connections. k_3 is a penalty factor to calibrate the relative importance of $a_1(t)$ and $a_2(v)$.

Our experimentation indicates that the model is not very sensitive to different values of k_1 , k_2 and k_3 . In the example mentioned later, an

increase in travel time is penalized twice as much as waiting time or early departure; i.e. $k_3 = 2k_1 = 2k_2$.

Since the dissatisfaction, as measured in (3.1), is a function of the passengers' desired departure time and the route, the daily demand for a given origin-destination pair is partitioned into subgroups corresponding to different times of the day. All passengers of a subgroup are assumed to have the same desired departure time equal to the average desired departure time for the subgroup. In the example mentioned later, the number of subgroups is equal to the number of routes available from the origin to the destination. The passengers subgroup associated with a given route is the set of passengers for which this is the most attractive route. Two adjacent subgroups are defined by the moment in time where the two routes associated with them have equal attraction. Referring to Figure 1, at time A defining subgroups G_1 and G_2 , routes u_1 and u_2 have equal attraction. This approach

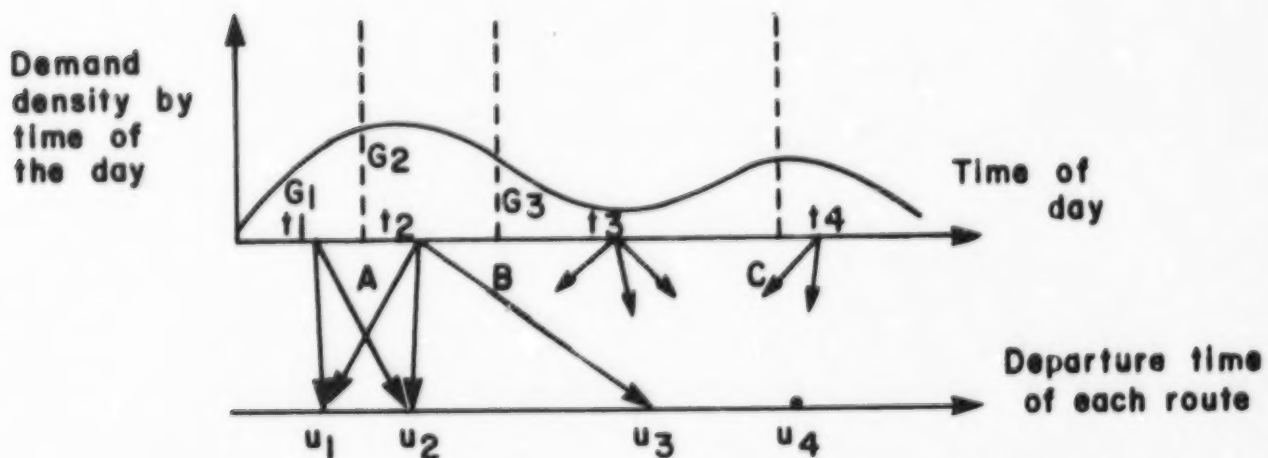


FIGURE 1

THE DIVISION OF PASSENGERS IN SUBGROUPS

implies that the attractiveness term is evaluated more precisely for origin-destination pairs serviced by a larger number of routes. In Figure 1, region G_i represents a passengers subgroup having the desired (average) departure time t_i .

A cost $p(t_i, j)$ is associated with each passenger of group G_i using route j (leaving at time u_j). Note that in our example the passengers of a subgroup may be assigned only to the most attractive route or to one of the two adjacent ones.

To simplify the notation in the rest of this paper, each subgroup of passengers associated with a given origin-destination pair is referred to as a market. The set of all these markets is denoted by Δ .

Assume that t_m is the average desired departure time for the passengers of market m . These passengers dissatisfaction is computed as follows:

$$(3.2) \quad \sum_{r \in Q_m} h_r^m p(t_m, r)$$

where Q_m is the set of routes serving the market m , and h_r^m is the average number of passengers of market m assigned to route r .

3.2. Availability factor

Before analyzing this factor, the concept of "passenger overload" (PO) is introduced

3.2.1. Concept of "passenger overload" on a flight-leg

To evaluate a flight schedule, it is assumed that the daily demand for each origin-destination pair is a random variable. Furthermore, the airline works with average demand and, generally, uses the concept of the load factor to take into account the daily variability of the demand.

The load factor on a flight-leg is the percentage of aircraft capacity (80%, for instance) assumed available for planning purposes. Hence the problem becomes deterministic, and, in models based on this principle, the passengers have to be reassigned to other routes as soon as their average number exceeds the number of seats specified by the load factor. In Figure 2, the dotted line indicates the number of passengers in excess (passenger overload) on an aircraft of capacity equal to 100 if the load factor is 80%

Given a daily demand distribution for each flight-leg, our model determines the passenger overload on each flight-leg. This PO is the expected number of passengers in excess of the aircraft capacity given the distribution of the number of passengers assigned to the flight-leg. The approach is illustrated in Figure 2 where the curved line indicates the passenger overload determined by the model.

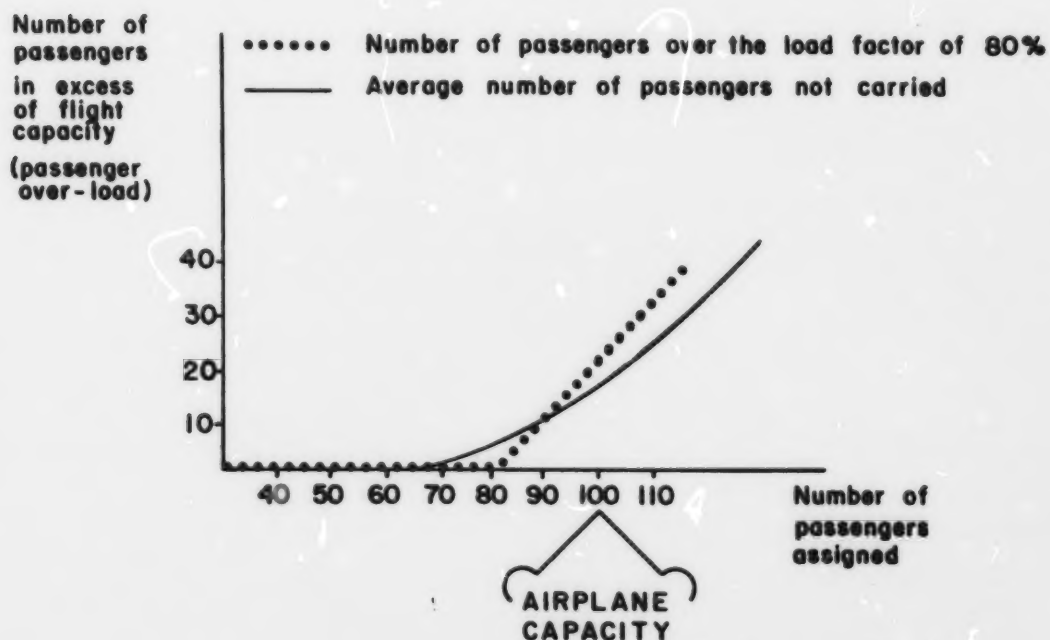


FIGURE 2
PASSENGERS IN EXCESS

An analysis of three months data for the Air Canada network shows that the distribution of the daily number of passengers using a flight can be approximated by a normal distribution. This approximation gives a very good estimate of the expected passenger overload on a flight-leg.

Let $Z_a(f_a) = E(c_a, f_a, \sigma_a)$ be the expected passenger overload on flight-leg a using an aircraft of capacity c_a , assuming that the average number of passengers assigned to a is f_a , and that the standard deviation of the number of passengers assigned to a is σ_a . The coefficient of variation of the demand λ_a for flight-leg a is a function of the coefficients of variation of the demands for the origin-destination pairs using this leg and the correlations between these pairs. Details of this evaluation are given in Soumis (1979). Since the number of passengers assigned to the leg is a fraction of the demand for flight-leg a , it follows that it has the same coefficient of variation λ_a , and hence $\sigma_a = \lambda_a f_a$. If $E(c_a, f_a, \sigma_a)$ is written in terms of the standardized $N(0,1)$ variable, then

$$(3.3) \quad Z_a(f_a) = E(c_a, f_a, \sigma_a) = E(c_a, f_a, \lambda_a f_a) = \lambda_a f_a E\left(\frac{c_a - f_a}{\lambda_a f_a}, 0, 1\right)$$

Let

$$\rho = \frac{c_a - f_a}{\lambda_a f_a}$$

$$E(\rho, 0, 1) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}\rho^2} + \rho \int_{\rho}^{\infty} (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2} dx$$

In the example, the following relation is an excellent approximation:

$$E(p,0,1) = \begin{cases} .4 \left(1 - \frac{p}{5}\right)^7 & \text{if } p \leq 5 \\ 0 & \text{otherwise.} \end{cases}$$

Table 1 compares $E(p,0,1)$ with its approximation. The variation of p from ∞ to 0 corresponds to an assigned number of passengers from 0 to the capacity of the aircraft.

p	∞	10	5	4	3	2	1.5	1	.5	0	-.5
$E(p,0,1)$	0	.0000	.00001	.0001	.0005	.0086	.029	.083	.198	.399	.70
$.4\left(1 - \frac{p}{5}\right)^7$	(0)	(0)	.0000	.00005	.0006	.0111	.032	.083	.191	.400	.77

TABLE 1
COMPARISON BETWEEN $E(p,0,1)$ AND ITS APPROXIMATION

3.2.2. Loss of revenue due to passenger overload

Since a given flight-leg may be used by passengers from several origin-destination pairs, and the cost of losing a passenger may vary from one pair to another, it becomes necessary to determine the proportion of the total passenger overload belonging to each origin-destination pair. In order to do this, the following two hypotheses are made:

First Hypothesis:

The probability that a passenger assigned to a flight-leg will be able to use it is independent of the origin-destination pair to which he belongs.

Notice that this hypothesis is correct if the seats on a flight-leg are attributed on a first-come first-served basis and if the reservations are made randomly for each origin-destination pair. But this hypothesis is not valid if priority is given to some origin-destination pairs or if blocks of seats are reserved for some origin-destination pairs.

Under this hypothesis, the passenger overload on flight-leg a of route r due to passengers of market m being assigned to route r is:

$$\frac{h_r^m}{f_a} Z_a(f_a)$$

where

$$f_a = \sum_{m \in \Delta} \sum_{r \in E_a^m} h_r^m,$$

is the total number of passengers assigned to flight-leg a , where E_a^m is the set of routes for market m using flight-leg a , and where

$\frac{h_r^m}{f_a}$ specifies the portion of passengers of market m on flight leg a .

Second Hypothesis:

For an optimal assignment of passengers, the probability that the same passenger be in excess of capacity on more than one flight-leg making up the route is equal to zero.

This hypothesis is not very restrictive, and it simplifies greatly the statement of the model. It is not very constraining because the most important origin-destination pairs are served by direct flight, and the hypothesis does not apply. Furthermore, an airline is looking for a solution where the probability for the passenger to be in excess of capacity is very small. Since the magnitude of the probability that the same passenger is part of the overload on two different flight-legs is of the order of the square of this probability value, it becomes practically negligible. Notice also that we can control the effects of this hypothesis by inspection of the solution.

During the iterative process, some routes may become overloaded. Hence the probability for a passenger to be in excess of capacity on more than one flight-leg may be high. Since these terms are neglected, the cost for these routes increases, and the algorithm tends to reduce the number of passengers on these routes first.

Under the second hypothesis, the expected passenger overload on route r due to the market m is:

$$(3.4) \quad \sum_{a \in F_r} \frac{h_r^m}{f_a} Z_a(f_a) = h_r^m \sum_{a \in F_r} \frac{Z_a(f_a)}{f_a}$$

where F_r is the set of flight-legs making up route r . The cost of the expected passenger overload on route r due to market m is then:

$$(3.5) \quad B^m h_r^m \sum_{a \in F_r} \frac{Z_a(f_a)}{f_a}$$

where B^m is the revenue associated with passengers of market m (for example, the average ticket price).

The total loss of revenue for the airline due to the capacity constraints of the aircraft is obtained by taking the summation over all markets and all routes for each market:

$$(3.6) \quad \sum_{m \in \Delta} \sum_{r \in Q_m} B^m h_r^m \sum_{a \in F_r} \frac{Z_a(f_a)}{f_a}$$

3.3. Model Statement

The model can be summarized as follows:

$$(3.7) \quad \text{Min} \left\{ \sum_{m \in \Delta} \sum_{r \in Q_m} B^m h_r^m \sum_{a \in F_r} \frac{Z(f_a)}{f_a} \right\} + \left\{ \sum_{m \in \Delta} K_m \sum_{r \in Q_m} h_r^m p(t_m, r) \right\}$$

$$\text{Subject to: } \sum_{r \in Q_m} h_r^m = N_m, \quad m \in \Delta$$

$$h_r^m \geq 0, \quad m \in \Delta, \quad r \in Q_m.$$

where:

$$f_a = \sum_{m \in \Delta} \sum_{r \in E_a^m} h_r^m,$$

Δ is the set of markets,

Q_m is the set of routes serving the market m ,

K_m is a scalar for the market m ,

N_m is the demand for the market m ,

t_m is the average desired departure time for passengers of market m ,

B^m is the revenue for the passengers of market m ,

h_r^m is the average number of passengers of the market m assigned to route r ,

F^r is the set of flight-legs making route r ,
 f_a is the average number of passengers assigned to flight-leg a ,
 $Z(f_a)$ is the average passenger overload for flight-leg a ,
 E_a^m is the set of routes serving the market m that use flight-leg a ,
 $p(t_m, r)$ is a measure (in weighted unit of time) of the dissatisfaction of the passengers of market m assigned to route r .

The objective function is the sum of the loss of revenue due to passenger overload and the cost for passengers' dissatisfaction. The constraints specify that each passenger has to be assigned to a route.

Further explanation is required for K_m . This scalar is used to give a money value to dissatisfaction. For a given market m , K_m is chosen so that the cost of dissatisfaction becomes equal to B_m (the normal revenue for a market m passenger) when the dissatisfaction level makes the route unacceptable. In the example, the authors' own bias is used to determine K_m for different markets. The value of K_m is a linear decreasing function of the distance between the origin and the destination. This seems realistic since a passenger going further is, in general, more inclined to modify his travel plan. The technique used to determine K_m could be improved to integrate competition from other airlines to attract passengers of some markets.

The aircraft capacities may force some passengers to be reassigned to less attractive routes, but these routes must be acceptable to keep the dissatisfaction cost less than B^m . In fact, if such routes are not available, then the passengers are assigned to the most attractive route where they form part of the overload. The loss of revenue is at most B^m for each of them.

4. Solution technique

The choice of the solution technique is a function of the size of problem (3.7). In general, this is very large. For instance, the Air Canada network has 76 cities, 629 origin-destination pairs and more than 2 000 routes. The problem generated has more than 6 000 variables and 2 000 constraints if 3 routes are available for each market.

However, the structure of the problem is well suited to decomposition, and an efficient algorithm to solve the subproblems is available. For instance problems of the size mentioned in the preceding paragraph have been solved in 20 seconds C.P.U. on a Cyber 173. The solution obtained deviated from the optimal solution by less than one passenger per route.

4.1. Decomposition of the problem

The matrix of constraints in problem (3.7) has the following structure:

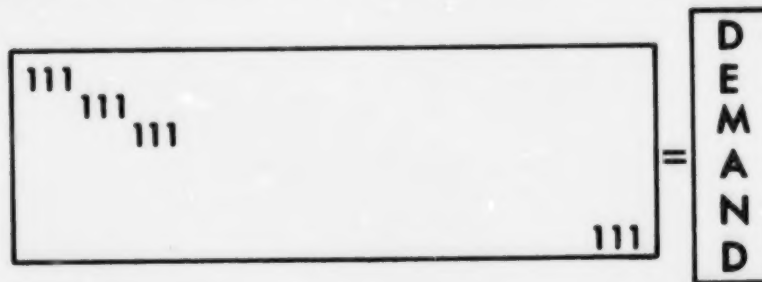


FIGURE 3: PASSENGER ASSIGNMENT PROBLEM CONSTRAINT MATRIX

Hence, the constraints for a market are independent of the constraints for the other markets. The interaction of passengers from different markets using the same flight-leg enters only into the objective function.

For problems in which the variables take on this type of block structure and where the objective function is strictly convex, one solution technique is to solve sequentially each subproblem corresponding to a block, keeping the value of the variables in the other blocks fixed. The solution is optimal when all the subproblems are solved without improving the value of the objective function. In the example with Air Canada network, each subproblem was solved 10 to 15 times.

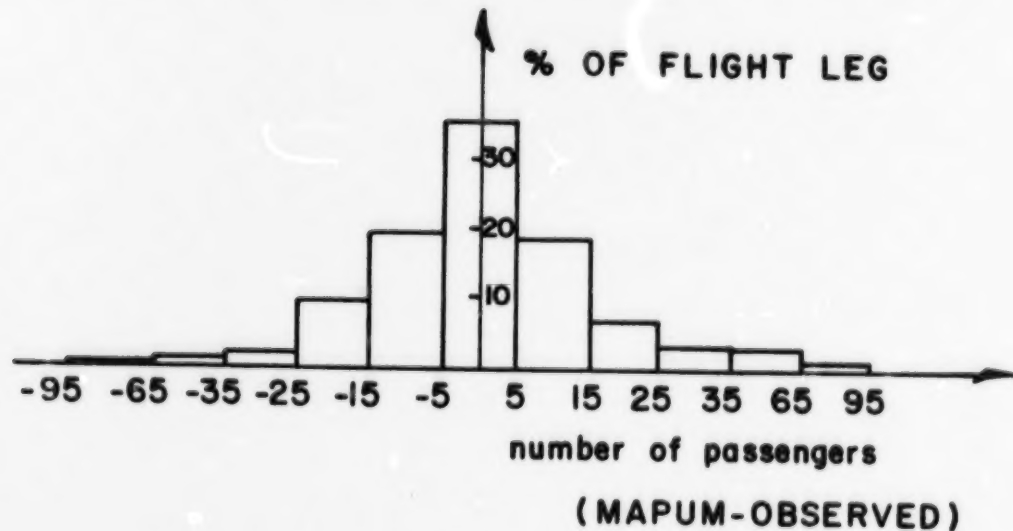
This solution technique has been used even if the objective function is not in general convex. (In Soumis (1979) the conditions for the convexity of the objective are studied.) However, note that the objective function is a sum of terms, each one of them corresponding to a flight-leg; only a few of these are not convex. Moreover, in all our numerical tests, we have experienced very stable solutions both with respect to the objective function and to the variables. This seems to indicate that the global optimum is attained.

Finally, notice that each subproblem corresponding to a block includes only one constraint. A very efficient technique to solve problems of this kind is given in appendix.

5. Numerical experimentation

The model has been tested with data from Air Canada. The experimentation was carried out using the observed demand and the operating schedule for the winter of 1976. Hence it is possible to compare the results generated by the model (MAPUM) with the observed distribution of passengers.

Figure 4 gives the distribution of the differences between the number of passengers assigned by MAPUM and the number of passengers observed on flight-legs, for the 539 flight-legs in the Air Canada schedule.



mean absolute difference = 11,6

FIGURE 4

DISTRIBUTION OF THE DIFFERENCE BETWEEN MAPUM AND OBSERVED
NUMBER OF PASSENGERS ON FLIGHT-LEGS

Figure 5 indicates the observed number of passengers versus the number of passengers assigned by MAPUM. Each point (x,y) in the figure is such that $x = (\text{Number of passengers observed on the flight-leg})/10$ and

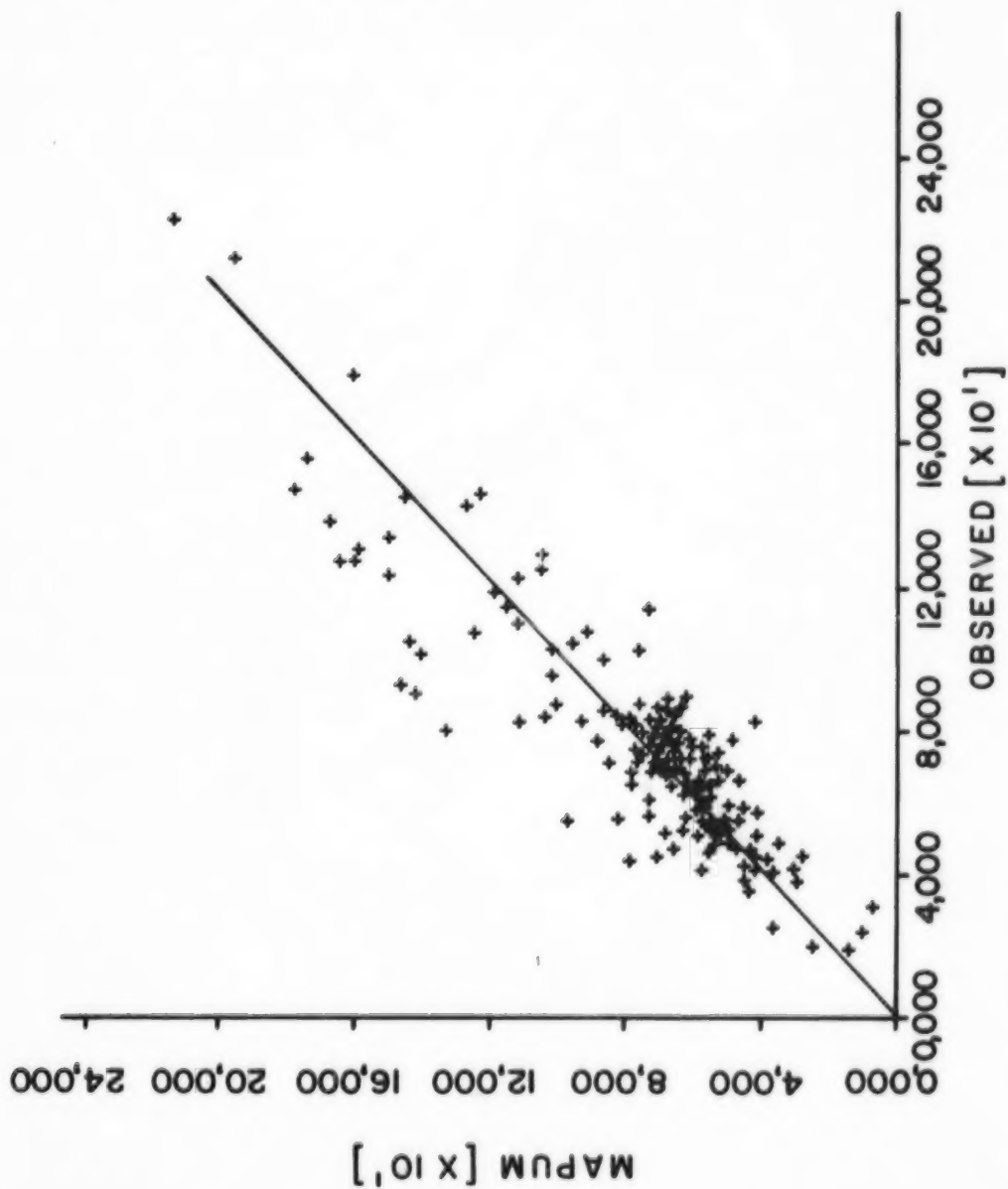


FIGURE 5

THE VALUE OF THE ESTIMATE OF
MAPUM IN WESTERN CANADA NETWORK

$y = (\text{Number of passengers assigned by MAPUM on the flight-leg})/10$. The estimate improves as the points approach the diagonal.

There are two sources of errors explaining the difference between the number of passengers observed and the number of passengers assigned by the model. A first source is the error in the estimate of the parameters of the model, and a second source is due to the model itself.

To illustrate the first source of error, note that the demand density curves by time of the day (Figure 1) provided by the company are not always very compatible with the observed departure times of the passengers. Also, the standard deviation in the daily number of passengers using a flight-leg has to be estimated. In the example, if these estimates are replaced by the observed standard deviations, the mean absolute error is decreased from 11.3 to 9.1

Our model places the emphasis on an adequate treatment of the effect of aircraft capacity limits rather than on an indepth analysis of the attractiveness factor. Hence, a finer analysis of the attractiveness factor would probably generate even better results. Some of the errors are due to particular situations and could be corrected appropriately. For instance, the model does not provide any mechanism to include the passengers' preference for larger aircraft (Boeing 747, L1011) and the results indicate an underuse of these.

These results are however considered quite acceptable by Air Canada and the industry in general. It is not expected that much better results could be easily attained.

4. Conclusion

This model provides a new approach to the passenger assignment problem based on the concept of passenger overload and uses mathematical programming techniques. The results show that, even using a somewhat crude approximation for the attractiveness factor, the solution is quite good. Moreover, the computer time needed to obtain a solution is very small. This approach is thus appropriate in several contexts. The model could be integrated into an automatic schedule generator (see Soumis, 1980), or it could be used in a man-machine process to evaluate and modify schedules.

The structure allows several modifications. The model described in this paper is a "system optimization" model in the sense that the passengers are assumed to behave to minimize total system cost. As opposed to this, a "user optimization" model could be used to determine an assignment where each passenger tries to maximize his own satisfaction. A similar solution technique could be used to solve this problem. The model could also be modified to integrate variable demands. The analysis of these modifications can be found in [24] but the data needed to test the approach are not yet available.

ACKNOWLEDGEMENTS

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APPENDIX

Solution technique for problems with one constraint

Each subproblem to be solved has the form:

$$\begin{aligned}
 (A.1) \quad & \text{Min} \quad g(x) \\
 & \text{Subject to: } \sum_{j=1}^n x_j = b \\
 & x_j \geq 0, \quad j=1,2,\dots,n.
 \end{aligned}$$

The solution technique is iterative. Given a feasible solution x^h , an improving direction is determined, and the objective function is minimized in this direction. Several alternative methods to determine the improving directions are analyzed in Soumis (1979) and the most efficient one for our example was to define the improving direction as the optimal solution of the following problem.

$$\begin{aligned}
 (A.2) \quad & \text{Min} \quad \sum_{j=1}^n c_j d_j + \frac{1}{2} a_j d_j^2 \\
 & \text{Subject to } \sum_{j=1}^n d_j = 0 \\
 & d_j \geq 0, \quad j \in J = \{j | x_j^k = 0\} \\
 & \text{where } c_j = \frac{\partial g(x^k)}{\partial x_j}, \quad a_j = \frac{\partial^2 g(x^k)}{\partial^2 x_j} > 0.
 \end{aligned}$$

In Soumis (1979) it is shown that all a_j are positive. The objective function of (A.2) is the second order approximation of $g(x)$ at x^k if $\frac{\partial^2 g(x^k)}{\partial x_i \partial x_j} = 0$ for all $i \neq j$. In the example, this hypothesis is valid in most cases since $\frac{\partial g(x^k)}{\partial x_i \partial x_j} \neq 0$ implies that two different routes for a given market use the same flight-leg. In any case, when it is not valid, a first order technique may be used instead. In Soumis (1979) it is shown that an optimal solution of (A.2) can be obtained analytically as follows:

i) Order the variables with index in J such that

$$c_{j_1} \leq c_{j_2} \leq \dots \leq c_{j_k}$$

ii) Evaluate iteratively:

$$\mu_t = \sum_{j \notin J} \frac{c_j}{a_j} + \sum_{i=1}^t \frac{c_{j_i}}{a_{j_i}} \quad \bigg/ \quad \sum_{j \notin J} \frac{1}{a_j} + \sum_{i=1}^t \frac{1}{a_{j_i}}$$

until either $c_{j_\tau} \leq \mu_\tau < c_{j_{\tau+1}}$, or $\tau = k$.

iii) Then an optimal solution of (4.2) is

$$d_j = \begin{cases} \frac{\mu_\tau - c_j}{a_j} & \text{if } j \notin J \text{ and if } j \in J \text{ and } c_j < \mu_\tau \\ 0 & \text{otherwise} \end{cases}$$

This direction is both very efficient and very easy to obtain. For problems with three variables, our experience shows that starting from the extreme initial solution $x_1=b$, $x_2=0$, $x_3=0$, the optimum solution is obtained after an average of 2 iterations in .02 second. In the actual solution of a passenger assignment problem, the average solution time for a subproblem is .012 second. The average number of iterations is reduced to 1.1 since in most cases, only one reoptimization of the subproblem is performed.

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